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On level spacing distributions for 2D non-normal Gaussian random matrices

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Abstract
The level spacing distribution of general, non-normal, Gaussian random 2D matrices is derived. In particular, tridiagonal matrices have no level repulsion and show a halfsided Gaussian distribution. General non-normal matrices show strong level repulsion. The repulsion exponent is $2 - 0_{\log}$.

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1. Introduction

Quantum systems are described by observables, which are represented by selfadjoint operators $A = A^+$. The corresponding matrix representation A_{ij} satisfies $A_{ij} = A^*_{ji}$. Physical symmetries are described by unitary operators $UU^+ = U^+U = 1$, with U_{ij} as the corresponding matrix. The eigenvalue distribution of statistical sets of selfadjoint, unitary or real symmetric operators has been studied intensively in the last few decades [1–5].

There are also other, more general matrices, which are of physical interest but not enjoying the particular property of being symmetric, Hermitian or unitary. The common property of these latter three classes of matrices is that they commute with their respective adjoint ones, i.e. $[A, A^+]_- = 0$ or $AA^+ = A^+A$. This common property guarantees that the eigenvectors are mutually orthogonal, $\varphi_a \perp \varphi_b$. Operators (matrices) with

$$[A, A^+]_- = 0 \tag{1}$$

are denoted as *normal* operators (matrices). If this property (1) does not hold,

$$[A, A^+]_- \neq 0, \tag{2}$$

the operator (matrix) is called *non-normal*, cf [7–10]. The question arises if some statements can be made about the eigenvalue distribution of non-normal operators or matrices?

The case of fully complex values for the matrix elements of a Gaussian ensemble of random matrices has been studied by Ginibre [1, 4, 6], and the cubic level repulsion was obtained. The general matrices play an important role in the quantum chaos of dissipative systems, i.e. those coupled to a heat bath [4]. Ginibre's method cannot be directly applied to the general real case. In this paper we calculate the level spacing distribution for a wide class of 2D real non-normal Gaussian random matrices, generalizing them also by considering different variances for the diagonal and off-diagonal elements, and find (almost) quadratic level repulsion, namely $\propto S^2 \log(1/S)$.

2. Non-normal matrices

Consider 2×2 matrices $A = (A_{ij})$, where $i, j = 1$ or 2. This matrix has two diagonal elements, which can always be chosen as a and $-a$. For general A_{ij} introduce $A_s = \frac{1}{2}(A_{11} + A_{22})$. Then $A_{11} - A_s \equiv a = -(A_{22} - A_s)$, i.e. one subtracts the diagonal matrix $A_s\underline{1}$ to get formula (3) without loss of generality. Quite generally, for a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ the level spacing $S = \lambda_1 - \lambda_2 = \sqrt{(a-d)^2 + 4bc}$ depends only on the difference $a - d$, so that we can arbitrarily shift a and d by a constant, in particular by A_s.

Let a be real and the nondiagonal elements be b_1 and b_2; thus

$$A = (A_{ij}) = \begin{pmatrix} a & b_1 \\ b_2 & -a \end{pmatrix}. \tag{3}$$

If also b_i are real and $b_1 = b_2$, the matrix A is symmetric. If b_i are complex and $b_2 = b_1^*$, the matrix A is selfadjoint or Hermitian. If *in addition* the relations hold $|b_1|^2 = 1 - a^2 = |b_2|^2$ and $a \leqslant 1$, A is unitary.

The eigenvalues of A follow from

$$|A - \lambda\underline{1}| = \begin{vmatrix} a-\lambda & b_1 \\ b_2 & -a-\lambda \end{vmatrix} = \lambda^2 - a^2 - b_1 b_2 = 0, \tag{4}$$

i.e.

$$\lambda_{1,2} = \pm\sqrt{a^2 + b_1 b_2}. \tag{5}$$

The eigenvalues are real for symmetric ($b_1 = b_2$), for Hermitian ($b_2 = b_1^*$) and for two-dimensional unitary ($a^2 + |b_1|^2 = 1$) matrices. In case of $b_1 b_2$ real and larger than $-a^2$ the eigenvalues are still real, in general not.

If the matrix A is *not* symmetric, Hermitian or unitary, it is no longer normal. In the general case, one finds for the commutator

$$[A, A^+]_{-} = \begin{pmatrix} |b_1|^2 - |b_2|^2 & 2a(b_2^* - b_1) \\ 2a(b_2 - b_1^*) & |b_2|^2 - |b_1|^2 \end{pmatrix}. \tag{6}$$

Apparently the commutator $[A, A^+]_{-} = 0$ is zero if $b_2 = b_1^*$, i.e. in the symmetric and the Hermitian cases. In general, 2×2 matrices A are non-normal, $[A, A^+]_{-} \neq 0$.

3. Real tridiagonal matrices

We first study the special case of a tridiagonal 2×2 matrix, setting $b_2 = 0$ and $b_1 \equiv b$, real:

$$A = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}, \text{ real, upper tridiagonalmatrix.} \tag{7}$$

The eigenvalues are $\lambda_{1,2} = \pm|a| = \pm a$. The eigenvectors, properly normalized, are

$$\varphi_1 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \varphi_2 = \frac{1}{\sqrt{4a^2+b^2}}\begin{pmatrix}-b\\2a\end{pmatrix}. \tag{8}$$

The angle α between the eigenvectors is not $\pi/2$ as for normal matrices with different eigenvalues, but is

$$\cos\alpha \equiv \langle\varphi_1|\varphi_2\rangle = (1\,0)\begin{pmatrix}-b\\2a\end{pmatrix}\cdot\frac{1}{\sqrt{4a^2+b^2}} = \frac{-b}{\sqrt{4a^2+b^2}}. \tag{9}$$

Special cases are the orthogonal case $b = 0$; thus $\cos\alpha = 0$ and $\alpha = \frac{\pi}{2}$ (eigenvectors are perpendicular, $\perp$). The parallel/antiparallel cases are obtained if $b \to \infty$. Thus $\cos\alpha = -\operatorname{sgn} b$, implying $\alpha = (2n+1)\pi$ (the eigenvectors are antiparallel $\rightleftharpoons$) or $\alpha = 2n\pi$ (the eigenvectors are now parallel, $\Rightarrow$); always $n = 0, \pm 1, \ldots$.

The matrix commutator reads

$$[A, A^+]_- = b\begin{pmatrix} b & -2a\\ -2a & -b\end{pmatrix} \neq 0 \qquad \text{if} \quad b \neq 0, \text{ non-normal case.} \tag{10}$$

The commutator itself is, of course, a Hermitian matrix, per construction.

Let us calculate the level spacing or distance S between the two eigenvalues, with $S \geqslant 0$:

$$S = |\lambda_1 - \lambda_2| = 2|a|. \tag{11}$$

For the matrix elements on the diagonal, a, we assume a Gaussian distribution as usual. The (normalized) distribution $g(b)$ of real b can be allowed to be arbitrary

$$g(a) = \frac{1}{\sigma\sqrt{\pi}}\cdot\exp\left(-\frac{a^2}{\sigma^2}\right), \qquad \int_{-\infty}^{+\infty} g(a)\,\mathrm{d}a = 1. \tag{12}$$

This gives the following probability distribution $P(S)$ for the level spacing S of a (non-normal) real tridiagonal matrix:

$$P(S) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}a\,\mathrm{d}b\,\delta(S-2|a|)g(a)g(b) = \int_{-\infty}^{+\infty}\mathrm{d}a\,\delta(S-2|a|)g(a). \tag{13}$$

Thus, completely independent of the b-distribution

$$P(S) = \frac{1}{2}g\left(\frac{S}{2}\right)\cdot 2 = \frac{1}{\sigma\sqrt{\pi}}\exp\left(-\frac{S^2}{4\sigma^2}\right), \qquad S \geqslant 0. \tag{14}$$

This is 'half' a Gaussian, since $S \geqslant 0$ is always positive, and it is normalized.

In this case of a non-normal real tridiagonal matrix there is no level repulsion, corresponding to a level repulsion exponent 0, and we have a non-repulsive Gaussian level spacing distribution. Clearly this as well holds for a lower instead of an upper tridiagonal matrix, and of course, the result is the same as for the diagonal matrices.

4. Physical example of non-normal matrix

There are many cases in which non-normal matrices appear in physics. The prominent ones are known from fluid mechanics [8, 9]. They are also met in other fields, including ecology, cf [11]. As an example for illustration we here consider the classical, damped, harmonic oscillator $m\ddot{x} = -cx - \alpha\dot{x}$, where x is its amplitude; m, c and α are its mass, string and damping constants, respectively. We nondimensionalize by measuring the amplitude in terms of the initial amplitude x_0 and the time in terms of the inverse bare eigenfrequency $\omega_0 = \sqrt{c/m}$. The set of first-order equations of motion then is $\dot{x} = v$ and $\dot{v} = -x - 2\gamma v$. Here, $2\gamma = \frac{\alpha}{m\omega_0}$

is the damping rate in terms of the bare eigenfrequency of the oscillator. In a matrix form, the equations of motion are

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2\gamma \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \equiv L \begin{pmatrix} x \\ v \end{pmatrix}. \tag{15}$$

The matrix L is real and non-symmetric. Its nondimensionalized eigenvalues are $\lambda_\pm = -\gamma \pm \sqrt{\gamma^2 - 1}$. These eigenvalues are real in the overdamped case $\gamma > 1$, they are conjugate complex in the underdamped case $\gamma < 1$, and they are degenerate (and real) in the aperiodic limit $\gamma = 1$. Note that the nondimensionalized bare eigenfrequency is 1. The linear matrix operator L (15) of the damped harmonic oscillator is non-normal,

$$[L, L^+]_- = -4\gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{16}$$

It is only for the ideal, undamped oscillator $\gamma = 0$ that the dynamical matrix is normal. As soon as there is nonzero damping $\gamma \neq 0$, the linear operator L for the damped harmonic oscillator dynamics is non-normal.

The normalized eigenvectors are

$$\Phi^0_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \lambda_\pm \end{pmatrix}, \qquad \text{underdamped}, \tag{17}$$

and

$$\Phi^0_\pm = \frac{1}{\sqrt{1 + \lambda_\pm^2}} \begin{pmatrix} 1 \\ \lambda_\pm \end{pmatrix}, \qquad \text{overdamped}. \tag{18}$$

The angle $\angle$ between the eigenvectors is calculated as $\left| \langle \Phi^0_+ | \Phi^0_- \rangle \right| = \cos(\angle) = \gamma$ in the underdamped case $\gamma < 1$ and $=1/\gamma$ in the overdamped case $\gamma > 1$. In the special case of the aperiodic limit $\gamma = 1$, we find $\Phi^0_+ = \Phi^0_- = \frac{1}{\sqrt{2}}(1, -1)^+$. The time dependence of the amplitude then is $x(t) = [1 + t(v + 1)]\,\mathrm{e}^{-t}$. In general, the amplitude oscillates. If $0 \leqslant \gamma < 1$ the eigenvector Φ^0_- with increasing γ rotates towards Φ^0_+, if $\gamma = 1$ both coincide, and if $1 < \gamma \to \infty$ the eigenvector Φ^0_- rotates back into the perpendicular position to Φ^0_+.

5. Real matrix with positive nondiagonal elements

We now consider the more general case of a real 2×2 matrix with positive definite nondiagonal elements

$$A = \begin{pmatrix} a & |b_1| \\ |b_2| & -a \end{pmatrix}, \qquad a \text{ real}, b_{1,2} \text{ real}. \tag{19}$$

This matrix is non-normal, unless $b_1 = b_2$. Its real eigenvalues always read

$$\lambda_\pm = \pm\sqrt{a^2 + |b_1||b_2|}. \tag{20}$$

Their distance $S = |\lambda_+ - \lambda_-|$ is given by the positive root

$$S = 2\sqrt{a^2 + |b_1||b_2|}, \qquad S \geqslant 2a \text{ always}. \tag{21}$$

Let now the three real numbers a, b_1, b_2 be Gaussian distributed according to (12). All are centred around zero; they have widths σ for a and $\sigma_{1,2}$ for $b_{1,2}$. The distribution $P(S)$ of the level distances S is given by

$$P(S) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}a\, \mathrm{d}b_1\, \mathrm{d}b_2\, \delta\left(S - 2\sqrt{a^2 + |b_1||b_2|}\right) g(a) g(b_1) g(b_2). \tag{22}$$

If in particular $\sigma_2 \to 0$, i.e. if $b_2 \to 0$, the previous case (14) is rediscovered. In another particular case, if $b_1 = b_2 \equiv b$, the matrix A is real and symmetric. Then take $g(b_2)$ as the delta-function $\delta(b_1 - b_2)$. If in addition $\sigma_1 = \sigma$, same widths, we find $g(a)g(b) \propto \exp\left(-\frac{a^2}{\sigma^2} - \frac{b^2}{\sigma^2}\right)$. One can introduce plane polar coordinates r and φ with $a^2 + b^2 = r^2$ and calculate the integrals in (22). The result is the well-known Wigner distribution

$$P(S) = \frac{S}{2\sigma^2} \exp\left(-\frac{S^2}{4\sigma^2}\right). \tag{23}$$

For $\sigma^2 = 1/\pi$ the distribution function reads $P(S) = \frac{\pi}{2} S\, \mathrm{e}^{-\frac{\pi}{4}S^2}$, the well-known level distribution (Wigner surmise) for real, symmetric, Gaussian random matrices, normalized to unit mean level spacing. There is linear level repulsion $\propto S$, meaning that the level repulsion exponent is 1. This result is indeed the exact 2D GOE result, and quite surprisingly differs only a little from the infinite dimensional limiting GOE result, cf chapter 4, especially 4.4, in Haake's book [4].

Let us now come back to the general case (22). The zeros of the delta function as a function of a are $a_\pm = \pm\sqrt{\frac{S^2}{4} - |b_1||b_2|}$. Expanding the argument of the delta function around $a(S)$, one obtains

$$P(S) = \iint_{|b_1||b_2| \leqslant S^2/4} g(b_1)g(b_2)\left[\frac{g(a_+)}{4|a_+|/S} + \frac{g(a_-)}{4|a_-|/S}\right]. \tag{24}$$

Since $g(a_+) = g(a_-)$ and $|a_+| = |a_-| = a(S) = \sqrt{\frac{S^2}{4} - |b_1||b_2|}$, we find for the level distribution the following basic expression:

$$P(S) = \frac{S}{2\sigma\sqrt{\pi}} \exp\left(-\frac{S^2}{4\sigma^2}\right) I(S), \tag{25}$$

with the integral factor

$$I(S) = \iint_{|b_1||b_2| \leqslant S^2/4} \mathrm{d}b_1\, \mathrm{d}b_2 g_1(b_1) g_2(b_2) \frac{\mathrm{e}^{\frac{|b_1||b_2|}{\sigma^2}}}{a(S)}. \tag{26}$$

The function

$$a(S) = \sqrt{\frac{S^2}{4} - |b_1||b_2|} \tag{27}$$

is the zero of the delta function in equation (22) for positive S.

We again cross-check by considering the special choice $g_2(b_2) = \delta(b_2)$ and recover our previous result for the upper tridiagonal matrix: We now have $a(S) = S/2$, the constraint of the integral is satisfied automatically; thus $I(S) = 2/S$, immediately leading to $P(S) = \frac{1}{\sigma\sqrt{\pi}} \exp(-\frac{S^2}{4\sigma^2})$, which agrees with equation (14).

In the general case, we introduce new coordinates x_1, x_2 instead of $|b_1|, |b_2|$ in order to take easier care of the constraint, given by the border of the integral $I(S)$. First, due to evenness in b_1 and b_2 we can reduce the integration over the full coordinate plane to a four times the integral over the positive quadrant $b_1 \geqslant 0, b_2 \geqslant 0$. Then, define $x_i = 2b_i/S, i = 1, 2$, with $x_i \geqslant 0$, since b_i are restricted to positive values now. Then, we can write

$$I(S) = \frac{2S}{\pi\sigma_1\sigma_2} F(S), \tag{28}$$

with the factor

$$F(S) = \iint_{x_1 \geqslant 0, x_2 \geqslant 0, x_1 x_2 \leqslant 1} \mathrm{d}x_1\, \mathrm{d}x_2 \frac{\exp\left(-\frac{S^2}{4}\left[\frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2} - \frac{x_1 x_2}{\sigma^2}\right]\right)}{\sqrt{1 - x_1 x_2}}. \tag{29}$$

Introducing again new (hyperbolic) coordinates $u = x_1x_2 \in [0, 1]$ and $v = x_1/x_2 \in [0, \infty)$, with the Jacobian determinant $J = 1/(2v)$, we obtain

$$F(S) = \int_0^1 \frac{\mathrm{d}u\, \mathrm{e}^{\beta u}}{\sqrt{1-u}} \int_0^\infty \frac{\mathrm{d}v}{2v} \exp\left(-\beta u \left(\alpha_1 v + \frac{\alpha_2}{v}\right)\right), \tag{30}$$

where

$$\beta = \frac{S^2}{4\sigma^2}, \qquad \alpha_1 = \frac{\sigma^2}{\sigma_1^2}, \qquad \alpha_2 = \frac{\sigma^2}{\sigma_2^2}. \tag{31}$$

The v-integral can be expressed in terms of the modified Bessel function of second kind and zero order K_0 (cf [12] No. 3.478, 4), so that we have

$$F(S) = \int_0^1 \frac{\mathrm{d}u\, \mathrm{e}^{\beta u}}{\sqrt{1-u}} K_0(2\beta\sqrt{\alpha_1\alpha_2}u). \tag{32}$$

With the substitution $\xi = \sqrt{1-u}$, this integral can be rearranged into the alternative form

$$F(S) = 2\mathrm{e}^{\beta} \int_0^1 \mathrm{d}\xi\, \mathrm{e}^{-\beta\xi^2} K_0(2\beta\sqrt{\alpha_1\alpha_2}(1-\xi^2)). \tag{33}$$

As we could not simplify either of these expressions further in a closed form, we instead use it to consider the three limiting cases: (i) small $\beta \ll 1$, (ii) large $\beta \gg 1$ and (iii) large $\lambda = 2\beta\sqrt{\alpha_1\alpha_2} \gg 1$. Physically these limits mean: (i) $S \ll 2\sigma$, small level distances in comparison to the width of the diagonal element distribution. (ii) The level spacings are much larger than the width of the diagonal element distribution, $S \gg 2\sigma$, i.e. the large spacings on the tail of the level distribution are considered. (iii) The level distance is compared with the fluctuations of the off-diagonal elements b_i, i.e. $S \gg \sqrt{2}\sigma_g$, S being large compared with the geometric mean $\sigma_g \equiv \sqrt{\sigma_1\sigma_2}$ of the widths of b_i.

5.1. *Small β or $S \ll 2\sigma$*

In this case we observe ([12], No. 8.447, 1 and 3, and 8.362, 3) that $K_0(z)$ can be expanded as $K_0(z) = -\ln\frac{z}{2} - C + O(z)$, where C is the Euler constant 0.577215 . . . . This approximation leads to the result

$$F(S) = -\ln\frac{S^2}{4\sigma_1\sigma_2} - \gamma - 2C, \tag{34}$$

where $\gamma = \int_0^1 \frac{du \ln u}{\sqrt{1-u}} = -1.22741$. Thus the leading order for $\beta \propto S^2 \to 0$ is $F(S) = 2\ln(4\sigma_1\sigma_2 S^{-2})$, and with equations (25), (28) we have

$$P(S) = \frac{S^2}{\sigma\sigma_1\sigma_2\pi^{3/2}}\left(\ln\frac{4\sigma_1\sigma_2}{S^2} - \gamma - 2C + O(S^2)\right), \tag{35}$$

where the additive constant is $-\gamma - 2C = 0.07298$. This behaviour extends within the interval of the order of $S \leqslant \sqrt{\sigma_1\sigma_2}$, whose length shrinks to zero with either σ_1 or σ_2, and beyond that border, when $S > \sqrt{\sigma_1\sigma_2}$, we find the slightly perturbed half-Gaussian distribution, as will be explained in subsection 5.3, equation (40). Instead of cubic level repulsion characteristic for the general complex Gaussian matrices treated by Ginibre [1, 4, 6], we have here for the real matrices with positive but general Gaussian distributed nondiagonal elements only $2 - O_{\log}$ for the level repulsion exponent. Unfortunately, in Ginibre's work the special case of real general matrices, including in particular the non-normal matrices, has not been worked out explicitly because this case presents severe technical difficulties. Hence the importance of our findings, even if so far, limited to the two-dimensional matrices. Since the level repulsion phenomenon is controlled and determined by the interaction of two neighbouring levels (triple level 'collisions' having the probability zero), we do believe that the level repulsion exponent that we find for dimension 2 holds true for any dimension of random matrices of the given class, namely for real non-normal matrices. We see that the restriction from complex to real non-normal matrices changes the level repulsion exponent from 3 to practically 2 (because the effect of the logarithm is so small).

5.2. Large β or $S \gg 2\sigma$, tail of distribution

We go back to equation (33) and employ the mean value theorem for this integral. It says that there is a value $\bar{\xi}$ in the ξ-integration interval $[0, 1]$ and thus another constant $\bar{u} = 1 - \bar{\xi}^2 \in [0, 1]$, $\bar{u}$ independent of the integration variable ξ but in general a function of $\beta, \alpha_1, \alpha_2$, i.e. of $S, \sigma, \sigma_1, \sigma_2$, such that we have

$$F(S) = 2\,\mathrm{e}^{\beta\bar{u}} K_0(2\beta\sqrt{\alpha_1\alpha_2}\bar{u}). \tag{36}$$

Since for large β (or S) the main contribution comes from $\xi \approx 1$, $\bar{u}$ will be close to zero. For large β, we use the asymptotic expansion ([12] No. 8.451, 6)

$$K_0(z) = \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}\left(1 + O\left(\frac{1}{z}\right)\right), \tag{37}$$

and from (25, 28, 36) we obtain

$$P(S) = \frac{2S}{\sigma\pi\sqrt{\bar{u}\sigma_1\sigma_2}} \exp\left[-\left(2\bar{u}\frac{\sigma^2}{\sigma_1\sigma_2} + (1-\bar{u})\right)\frac{S^2}{4\sigma^2}\right], \tag{38}$$

where $\bar{u}$ is a function depending only weakly on $S, \sigma, \sigma_1, \sigma_2$ and is close to zero, but certainly smaller than 1, so that the argument of the exponential function in (38) is always negative. Thus, the tail of the spacing distribution $P(S)$ for large S always decays as a Gaussian.

5.3. Small σ_1 and/or σ_2, or large $\lambda = 2\beta\sqrt{\alpha_1\alpha_2}$ or limit $S \gg \sqrt{2}\sqrt{\sigma_1\sigma_2}$

If σ_1 or/and σ_2 go to zero, we have the limit of the triangular matrix with the 'half' Gaussian level spacing distribution (14). We wish to work out this limiting case from the general expression (32). When σ_1, σ_2 are small, or equivalently α_1, α_2 large, the argument λu of K_0, where $\lambda = 2\beta\sqrt{\alpha_1\alpha_2} = 2\beta\sigma^2/(\sigma_1\sigma_2)$, in (32) is very large, and thus the functional value of K_0 is exponentially small. In such a case the main contribution to the integral over the interval $[0, 1]$ stems from the values of u close to zero, and thus we can Taylor expand $\mathrm{e}^{\beta u}/\sqrt{1-u}$ in powers of u, namely $\mathrm{e}^{\beta u}/\sqrt{1-u} = 1 + \left(\beta + \frac{1}{2}\right)u + \left(\frac{3}{8} + \frac{\beta}{2} + \frac{\beta^2}{2}\right)u^2 + O(u^3)$, and also extend the upper integration limit from 1 to ∞, thereby committing an exponentially small error. The resulting integrals $\int_0^\infty u^m K_0(\lambda u)\,\mathrm{d}u$ for $m = 0, 1, 2, \ldots$ can be evaluated explicitly (cf [12] No. 6.561, 16). We get $\pi/(2\lambda)$, $1/\lambda^2$ and $\pi/(2\lambda^3)$, for $m = 0, 1, 2$ respectively, and so we obtain

$$F(S) = \frac{\pi}{2\lambda} + \frac{\beta + \frac{1}{2}}{\lambda^2} + \frac{\pi}{2\lambda^3}\left(\frac{3}{8} + \frac{\beta}{2} + \frac{\beta^2}{2}\right) + O(\lambda^{-4}). \tag{39}$$

Taking into account equations (25) and (28), we derive from (39)

$$P(S) = \frac{1}{\sigma\sqrt{\pi}} \exp\left(-\frac{S^2}{4\sigma^2}\right)\left(1 + \frac{2\beta+1}{\pi\lambda} + \frac{1}{\lambda^2}\left(\frac{3}{8} + \frac{\beta}{2} + \frac{\beta^2}{2}\right) + O(\lambda^{-3})\right) \tag{40}$$

which in the limit $\sigma_1\sigma_2 \to 0$, implying $\lambda \to \infty$, indeed goes to the result for the triangular matrices, namely the 'half' Gaussian distribution (14). Due to the extension of the upper integration limit from 1 to ∞ we have also lost some exponentially small terms of order $\exp(-\text{const}\cdot\lambda)$ in the third factor in equation (40), which are of course for large λ much smaller than the algebraic terms given there. Clearly, the limit of the triangular matrix is obtained if either σ_1, or σ_2, or both ($\rightarrow$ diagonal matrix) go to zero.

In the context of quantum chaos and energy level statistics, in all cases the final values of σ, σ_1 and σ_2, preferably σ, are chosen such that the level spacing distribution has a normalized (unit) first moment $\int_0^\infty SP(S)\,\mathrm{d}S = 1$, meaning that the mean level spacing is 1. We do without such normalization here, because the physical interpretation of the non-normal matrix has not been discussed in the corresponding quantum context. We just remark that the general Gaussian random matrices (with the same variance for the off-diagonal elements and twice larger for the diagonal matrix elements), including the non-normal matrices, with complex matrix elements and complex eigenvalues, play an important role in dissipative quantum systems, i.e. those quantal Hamiltonian systems which are coupled to a heat bath. For a detailed discussion and theory, see chapter 9 in Haake's book [4].

All the results of section 5 can be extended to the case of purely negative off-diagonal matrix elements, which is obvious from the correspondingly rewritten equations (19), (20) and (22).

The general case of arbitrary but possibly mixed positive and negative off-diagonal matrix elements in equation (3) is difficult to handle and remains an open problem. The case of fully complex values for the matrix elements of the Gaussian ensemble of random matrices has been studied by Ginibre [1, 4, 6], and cubic level repulsion was obtained. The reduction of that method to the general real case remains an open problem.

6. Relation to GOE ensemble

In this section, we want to clarify the relation between the distribution of our non-normal matrix ensemble and that of the traditional two-dimensional GOE ensemble, and show that in the real symmetric case they are indeed the same.

For a general two-dimensional real symmetric matrix, $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$, the level spacing is equal to $S = 2\sqrt{(a-c)^2 + 4b^2}$. Let in general all matrix elements a, b and c have the normalized Gaussian distribution $g_a(a)$, $g_b(b)$ and $g_c(c)$, as defined in (12), but generally with different variances σ_a^2, σ_b^2 and σ_c^2, respectively. The level spacing distribution is given as

$$P(S) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c\; g_a(a)g_b(b)g_c(c)\delta(S - \sqrt{(a-c)^2 + 4b^2}). \tag{41}$$

Introducing new coordinates $x = \frac{1}{2}(a+c)$, $y = \frac{1}{2}(a-c)$, with the Jacobi determinant $J = 2$, we see that the delta function does not depend on x. By setting $\sigma_a = \sigma_c$ (as necessary for the GOE ensemble) and performing the integration over x, we find

$$P(S) = \frac{\sqrt{2}}{\pi\sigma_a\sigma_b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}y\,\mathrm{d}b\,\mathrm{e}^{-\left(\frac{2y^2}{\sigma_a^2}+\frac{b^2}{\sigma_b^2}\right)}\delta(S - 2\sqrt{y^2 + b^2}). \tag{42}$$

Now, again, we take into account the GOE hypothesis, namely $\sigma_a^2 = \sigma_c^2 = 2\sigma_b^2 = 2\sigma^2$, i.e. the variance for the diagonal matrix elements is twice the variance of the off-diagonal matrix elements, and obtain (in terms of the off-diagonal variance σ^2)

$$P(S) = \frac{1}{\pi\sigma^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}y\,\mathrm{d}b\,\mathrm{e}^{-\left(\frac{y^2+b^2}{\sigma^2}\right)}\delta(S - 2\sqrt{y^2 + b^2}), \tag{43}$$

which is indeed exactly our special case, the exact two-dimensional GOE case, calculated and discussed just following equation (22). Therefore, indeed, our ensemble $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ with

$\sigma_a = \sigma_b = \sigma$ is precisely equivalent to the exact two-dimensional GOE ensemble, with the same variance σ^2 for the off-diagonal elements and twice larger variance $2\sigma^2$ for diagonal elements. This small and a little bit confusing difference in the two definitions is the consequence of our choice $c = -a$.

7. Asymmetric distribution of diagonal and nondiagonal elements

Now we go back to just such a real symmetric Gaussian ensemble $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ and generalize it by allowing for different widths in the statistics of the diagonal elements a and the nondiagonal elements b. Thus, we allow for different variances $\sigma_a \neq \sigma_b$. Then our starting equation for the level spacing distribution is (42),

$$P(S) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}a\,\mathrm{d}b\, g_a(a) g_b(b) \delta(S - 2\sqrt{a^2+b^2}). \tag{44}$$

The Gaussians $g_a(a)$ and $g_b(b)$ are normalized as in equation (12), but $\sigma_a \neq \sigma_b$ is admitted. Introducing polar coordinates $a = r\cos\varphi, b = r\sin\varphi, \varphi \in [0, 2\pi)$, and integrating over $r \in [0, \infty)$, we immediately get

$$P(S) = \frac{S}{4\pi\sigma_a\sigma_b}\int_0^{2\pi} \mathrm{d}\varphi \exp\left(-\frac{S^2}{4}\left(\sigma_a^{-2}\cos^2\varphi + \sigma_b^{-2}\sin^2\varphi\right)\right). \tag{45}$$

Further, using $\cos^2\varphi = \frac{1}{2}(1+\cos 2\varphi)$ and $\sin^2\varphi = \frac{1}{2}(1-\cos 2\varphi)$, and after some elementary manipulations we get, using the integral representation for $I_0(z)$,

$$P(S) = \frac{S}{2\sigma_a\sigma_b}\,\mathrm{e}^{-\frac{S^2}{8}(\sigma_a^{-2}+\sigma_b^{-2})} I_0\left(\frac{S^2}{8}\left(\sigma_a^{-2} - \sigma_b^{-2}\right)\right). \tag{46}$$

Here $I_0(z)$ is the modified Bessel function of the first kind and zero order (cf [12], No. 8.406). We now introduce the notations

$$\sigma_m^{-2} = \tfrac{1}{2}\left(\sigma_a^{-2} + \sigma_b^{-2}\right) \qquad \text{and} \qquad u = \tfrac{1}{8}\left(\sigma_a^{-2} - \sigma_b^{-2}\right). \tag{47}$$

While σ_m is a measure for the mean statistical width of both, the diagonal and the nondiagonal elements, the parameter u indicates the difference of the respective widths on the diagonal and nondiagonal. This difference parameter u can be positive or negative. $u > 0$ means that the nondiagonal element distribution is broader than that of the diagonal elements, while for $u < 0$ the statistical spread on the nondiagonal is smaller than on the diagonal. With these parameters, we can write

$$P(S) = \sqrt{1 - \left(4\sigma_m^2 u\right)^2}\,\frac{S}{2\sigma_m^2}\,\mathrm{e}^{-S^2/4\sigma_m^2} I_0(uS^2). \tag{48}$$

Note that the square root is always real, since always $\left|4\sigma_m^2 u\right| = \frac{|\sigma_a^{-2}-\sigma_b^{-2}|}{\sigma_a^{-2}+\sigma_b^{-2}} \leqslant 1$. The level distribution $P(S)$ is even in the difference parameter u, because $I_0(z)$ is an even function of z. Thus, the two cases of either broader or smaller statistical width on the diagonal relative to the nondiagonal elements behave the same. To elucidate limiting cases, we observe that $I_0(z)$ has the power (Taylor) expansion at small z:

$$I(z) = 1 + \frac{z^2}{4} + \frac{z^4}{64} + \frac{z^6}{2304} + O(z^8), \tag{49}$$

and the asymptotic expansion at large z is

$$I(z) \approx \frac{\mathrm{e}^{|z|}}{\sqrt{2\pi|z|}}. \tag{50}$$

Using these facts, it is easy to work out the following special and limiting cases of the general result (46).

7.1. $|u|S^2 \ll 1$, i.e. small S and/or small asymmetry $|u| \ll 1$

Small values of u mean that there are only small deviations from the two-dimensional GOE. In this case or in the limit $S \to 0$, we have

$$P(S) = \frac{S}{2\sigma_a\sigma_b}\,\mathrm{e}^{-\frac{S^2}{8}(\sigma_a^{-2}+\sigma_b^{-2})}\left(1+\frac{u^2S^2}{4}+\frac{u^4S^4}{64}+\frac{u^6S^6}{2304}+O(u^8S^8)\right). \qquad (51)$$

7.2. $|u|S^2 \gg 1$, outer tail of the distribution

The asymptotic expansion describes the case where S not only exceeds $\sigma_{a,b}$, but even exceeds $|u|^{-1/2} \geqslant \sigma_a, \sigma_b$. The behaviour in this outer tail of the distribution is

$$P(S) = \frac{1}{\sigma_m\sqrt{2\pi}}\sqrt{\frac{1-\left(4|u|\sigma_m^2\right)^2}{4|u|\sigma_m^2}}\,\mathrm{e}^{-\frac{S^2}{4\sigma_m^2}(1-4|u|\sigma_m^2)}, \qquad S \gg \frac{1}{\sqrt{|u|}} \geqslant \sigma_{a,b}. \qquad (52)$$

Thus the far outer tail is always Gaussian, in fact half-Gaussian because $S > 0$ is always only positive. But note that its width $\sigma_m\big/\sqrt{1-4|u|\sigma_m^2}$ in the far outer tail may be considerably larger than the average rms width σ_m.

Another way of presenting this asymptotic result is, after a straightforward calculation,

$$P(S) = \frac{1}{\sigma_{\max}\sqrt{\pi}}\,\mathrm{e}^{-\frac{S^2}{4\sigma_{\max}^2}}, \qquad \sigma_{\max} = \max\{\sigma_a, \sigma_b\}, \qquad S \gg \frac{1}{\sqrt{|u|}} \geqslant \sigma_{a,b}. \qquad (53)$$

7.3. Symmetric case $u = 0$, i.e. $\sigma_a = \sigma_b \equiv \sigma$

In this symmetric case $u = 0$ we have $\sigma_m = \sigma_a = \sigma_b = \sigma$, leading to

$$P(S) = \frac{S}{2\sigma^2}\,\mathrm{e}^{-\frac{S^2}{4\sigma^2}}, \qquad (54)$$

which for normalized level spacing $\langle S\rangle = 1$ and therefore $\sigma^2 = 1/\pi$ precisely is the Wigner surmise, the exact two-dimensional GOE case.

7.4. S arbitrary, but either $\sigma_a = 0$ or $\sigma_b = 0$

In this case, we either have a diagonal matrix with Gaussian distributed diagonal elements a or a pure nondiagonal but symmetric matrix with Gaussian distribution of its matrix elements b. Denoting the remaining nonzero width by σ, we have the following expansions for the control parameters σ_m and $|u|$ in terms of the vanishing width written as $\epsilon \to 0$:

$$\sigma_m^2 = 2\epsilon^2\left(1-\frac{\epsilon^2}{\sigma^2}\right), \qquad |u| = \frac{1}{8\epsilon^2}\left(1-\frac{\epsilon^2}{\sigma^2}\right), \qquad 4|u|\sigma_m^2 = 1-2\frac{\epsilon^2}{\sigma^2}. \qquad (55)$$

Since $|u| \to \infty$, we have to use equation (50) and find—but now without any restriction on S—from equation (52)

$$P(S) = \frac{1}{\sigma\sqrt{\pi}}\,\mathrm{e}^{-\frac{S^2}{4\sigma^2}}, \qquad (56)$$

valid for all $S > 0$, which is exactly the half-Gaussian (14).

7.5. Comments on the moments

Now we want to work out the moments of distributions (46) and (48). The zero-order moment or normalization is immediately satisfied as one proves from the integral representation (44). One can also check from (48), using [12], No. 6.643, 2,

$$\int_0^\infty P(S)\,\mathrm{d}S = 1. \tag{57}$$

For the first moment, the mean level spacing, one calculates

$$\langle S\rangle = \int_0^\infty S P(S)\,\mathrm{d}S = \frac{\sigma_m^3\sqrt{\pi}}{\sigma_a\sigma_b}\,{}_2F_1\left(\frac{3}{4},\frac{5}{4},1,r^2\right). \tag{58}$$

Here, r is the relative variance difference:

$$r = \left|\frac{\sigma_a^2-\sigma_b^2}{\sigma_a^2+\sigma_b^2}\right|. \tag{59}$$

Using the power expansion of the hypergeometric series ${}_2F_1\left(\frac{3}{4},\frac{5}{4},1,r^2\right)$ as a function of the relative variance difference r, we find

$$\langle S\rangle = \frac{\sigma_m^3\sqrt{\pi}}{\sigma_a\sigma_b}\left(1+\frac{15}{16}r^2+\frac{945}{1024}r^4+\frac{15015}{16384}r^6+O(r^8)\right). \tag{60}$$

In the special case of symmetric diagonal und nondiagonal element distributions $u = r = 0$ and $\sigma_m = \sigma_a = \sigma_b = \sigma$, we immediately obtain $\langle S\rangle = \sigma\sqrt{\pi}$. Thus the first moment is normalized to unity if $\sigma = 1/\sqrt{\pi}$, which is the well-known result leading to the Wigner surmise.

The general second moment is analytically much simpler, namely

$$\langle S^2\rangle = \int_0^\infty S^2 P(S)\,\mathrm{d}S = 2\int_0^\infty \mathrm{E}(S)\,\mathrm{d}S = \frac{4\sigma_a^2\sigma_b^2}{\sigma_m^2}, \tag{61}$$

where $\mathrm{E}(S)$ is the gap probability, i.e. the probability that on an interval of length S (after spectral unfolding!) there is no level. Evidently, after the normalization of the mean level spacing to 1, i.e. $\langle S\rangle = 1$, the relative variance $\mu^2 = \langle S^2\rangle - 1$ of the level spacing distribution $P(S)$ is minimal, when $\frac{4\sigma_a^2\sigma_b^2}{\sigma_m^2} = \frac{4\sigma_m^2}{1-(4|u|\sigma_m^2)^2}$ is minimal, which for fixed inverse mean variance σ_m precisely happens for the symmetric case $u = 0$. Hence,

$$\mu^2 = 2\int_0^\infty \mathrm{E}(S)\,\mathrm{d}S - 1 = \langle S^2\rangle - 1 = 4\sigma^2 - 1 = \frac{4}{\pi} - 1 = 0.27324, \tag{62}$$

which is the two-dimensional GOE result (Wigner surmise). We conclude that the two-dimensional GOE case, corrresponding to $u = 0$, gives the strongest level repulsion among all generalized real symmetric Gaussian random matrix ensembles. Probably this result can be generalized to any dimension D, including the infinite matrices. This is our conjecture.

Another interesting limiting case is that of maximal difference between the variances of the diagonal and nondiagonal elements. One of the variances tends to zero, say $\sigma_a = \epsilon \to 0$; the other one stays finite $\sigma_b = \sigma$. Then the difference parameter $u = 1/(8\epsilon^2) \to \infty$, the relative difference parameter $r \to 1$ becomes unity, and the inverse mean of the widths $\sigma_m^2 = 2\epsilon^2 \to 0$ also vanishes. One either can calculate the moments from equations (58) and (61) using these limits of the corresponding parameters, or can use the level statistics (56) directly. Both leads to the moments $\langle S\rangle = 2\sigma/\sqrt{\pi}$ and $\langle S^2\rangle = 2\sigma^2$. For the relative variance of the level distribution in this most asymmetric case, we then obtain

$$\mu^2 = \frac{\langle S^2\rangle - \langle S\rangle^2}{\langle S\rangle^2} = \frac{2-\frac{4}{\pi}}{\frac{4}{\pi}} = \frac{\pi}{2} - 1 = 0.5708. \tag{63}$$

We mention that the second moment of $P(S)$, after the normalization $\langle S \rangle = 1$, namely $\mu^2 = \langle S^2 \rangle - 1$, is largest for Poissonian matrices, equal to 1. For the uniform spectrum (one-dimensional harmonic oscillator) it is the smallest possible, equal to 0. For the two-dimensional GOE, it is $4/\pi - 1 = 0.27324$. Therefore, μ^2 seems to be a monotonic measure of the transition from Poisson to GOE and further to the uniform spectrum, which has been already observed in [13].

An important one parameter family of level spacing distributions which interpolates between the Poisson and GOE level spacing distributions is the Berry–Robnik formula [14–19], which has a sound physical foundation, and has been recently extensively studied and numerically verified to hold in the asymptotic (sufficiently deep) semiclassical limit of a sufficiently small effective Planck constant. The basic assumption behind that formula is the statistical independence (no correlations) between the regular and chaotic energy levels, which in turn rests upon the principle of uniform semiclassical condensation [15] (of the Wigner functions of the eigenstates). The deviations from the Berry–Robnik distribution function due to correlations (interacting levels due to tunnelling effects) between the regular and chaotic energy levels, at lower energies, or larger effective Planck constant, are a subject of intense current research [20].

8. Discussion and conclusions

Motivated by the important role of non-normal matrices in physics, we have discussed the statistical properties of their eigenvalues, in particular the level spacing distribution $P(S)$ for two-dimensional matrices, allowing for general matrices with real diagonal elements and positive real nondiagonal elements. In this case we have shown that the level repulsion exponent at small S is $2 - O_{\log}$, whilst the tail is always Gaussian. In the limit of small dispersion of the nondiagonal elements, we have recovered the 'half' Gaussian distribution which holds for tridiagonal matrices. These, surprisingly, have no level repulsion, analogous to the case of integrable systems with their Poissonian spacing distribution. Still they differ from a Poissonian but are fully Gaussian. Of course, in the special case of real symmetric matrices we again find the Wigner distribution, which is the exact (rigorous) 2D GOE level spacing distribution, and we explain the equivalence between the GOE ensemble and our real symmetric ensemble (with $c = -a$ for the diagonal matrix elements). The fully general case of entirely general real matrices with all elements Gaussian distributed (with different dispersions) is still open, whilst the fully general Gaussian complex case has been studied by Ginibre four decades ago [6], in which case the cubic level repulsion exponent has been found [1, 4], but the reduction to the real general case is still open. We believe that research in this direction is still important and interesting, with various interesting open questions, also in direction of mixed-type systems discussed at the end of section 7.

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Note added in proof. As Dr. Hans-Jürgen Sommers after reading our paper, section 5, has claimed, in the case of real non-diagonal elements $b_{1,2}$ with arbitrary signs, the level repulsion turns out to be $\propto S$, i.e., the repulsion index is 1. But this only holds if complex eigenvalues are allowed; for real eigenvalues only the index is still $2 - \mathrm{O}_{\log}$. We shall come back to this case and comment on it in the forthcoming paper.

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